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The role of dynamic modulation in the stability of viscoelastic flow down an inclined plane

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In this study we have theoretically investigated the effect of parallel superposition of modulation on the stability of single-layer Newtonian and viscoelastic flows down an inclined plane. Specifically, a spectrally based numerical technique in conjunction with Floquet theory has been used to investigate the linear stability of this class of flows. Based on these analyses we have demonstrated that parallel superposition of modulation can be used to stabilize or destabilize flow of Newtonian and viscoelastic fluids down an inclined plane. In general at low Reynolds number *Re* (i.e. $O(1)$) and in the limit of long and $O(1)$ waves the effect of dynamic modulation on the stability of viscoelastic flows is much more pronounced; however, relatively large modulation amplitudes are required to achieve significant stabilization/destabilization. In addition, the dependence of the most dominant modulation frequencies on *Re* and the Weissenberg number *We* have been identified. Specifically, it has been shown that for Newtonian flows low-frequency modulations are destabilizing and the most dominant modulation frequency scales with $1/Re$. On the other hand, for viscoelastic flows in the absence of fluid inertia low-frequency modulations are stabilizing and the most dominant modulation frequency scales with $1/We$. In finite-*Re* viscoelastic flows the most dominant destabilizing modulation frequency scales with $1/Re$ while the most stabilizing modulation frequency scales with $1/WeRe$. Finally, it has been demonstrated that the mechanism of both purely elastic and inertial instabilities in flows down an inclined plane is unchanged in the presence of dynamic modulation.

1. Introduction

The viscoelastic nature of most polymeric fluids can significantly affect flow instability in flows that are unstable due to capillary or inertial forces. In addition, viscoelastic forces can give rise to new mechanisms of instability that are absent in flow of Newtonian fluids (Shaqfeh 1996). In the past decade a number of studies have focused on elastic instabilities in simple shear flows such as those occurring in viscometric flows (Joo & Shaqfeh 1991, 1992*a, b*). These studies have identified various mechanisms for viscoelastic instabilities which hopefully can be used to provide insight into instabilities occurring in more complex geometries (Shaqfeh 1995; McKinley, Pakdel & Oztekin 1996).

Interfacial or free-surface instabilities have not received as much attention as bulk viscoelastic instabilities. However, recent studies (Ganpule & Khomami 1998, 1999*a, b*)

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have clearly demonstrated that the mechanism which gives rise to viscometric shear flow instabilities (i.e. coupling between normal stresses and streamline curvature) does not play a significant role in interfacial instabilities that occur in multilayer pressure or drag-driven channel flows. Moreover, the mechanism of many viscoelastic instabilities observed in free-surface flows such as in coating operations is not well understood. The free-surface flow of viscoelastic fluids is of great importance in many industrial applications when solids are coated with polymeric films and an understanding of the mechanism that drives this class of instabilities is needed.

The simplicity of the base flow kinematics in inclined plane flows, in combination with the fact that experimental evaluation of free-surface dynamics can be accurately made, makes this class of flows ideal for studying free-surface instabilities. As a result, the flow down an inclined plane has served as a paradigm in the investigation of Newtonian (Benjamin 1957; Yih 1963; Gupta 1967) and viscoelastic (Lin 1967; Shaqfeh, Larson & Frederickson 1989; Huang & Khomami 2000) free-surface instabilities.

The free-surface instability in flow down an inclined plane manifests itself in the form of travelling waves. The instability of long-wavelength disturbances has been studied by Yih (1963) and Benjamin (1957) for a Newtonian fluid and by Gupta (1957) for the upper-convected Maxwell (UMC) fluid, and it has been shown that the free surface can become unstable above a critical Reynolds number. In addition, it has been shown that reducing the inclination angle and increasing the surface tension (i.e. a minor influence in the limit of long-wave disturbances) increase the critical Reynolds number, while increasing the fluid elasticity decreases the critical Reynolds number.

The asymptotic technique used in the above studies is only applicable to long-wavelength disturbances. But in general the dominant mode of the instability can be due to disturbances of any wavelength. To study the effect of arbitrary disturbances, Lin (1967) constructed neutral stability diagrams for one-layer Newtonian flow down an inclined plane using a numerical approach and demonstrated that long-wave disturbances are the dominant mode of instability. Shaqfeh *et al.* (1989) performed a similar study for flow of an Oldroyd-B fluid down an inclined plane and demonstrated that even in the presence of fluid elasticity long waves are the most dangerous modes. Moreover, they demonstrated that elastic effects are always destabilizing in the limit of long waves but at intermediate wavenumbers they could be stabilizing at moderate Reynolds numbers. Recently, Huang & Khomami (2000) have systematically studied the free-surface and interfacial instabilities of multilayer viscoelastic flows down an inclined plane using both asymptotic and numerical methods and have demonstrated that free-surface and interfacial stability of this class of flows is a strong function of Reynolds and Weissenberg numbers, Re, We, and elasticity and viscosity stratification. In addition, the mechanism of free-surface instability of one-layer Newtonian and viscoelastice flows down an inclined plane has been investigated and it has been shown that in Newtonian flows the inertial instability is due to the perturbation shear stresses at the free surface (Kelley *et al.* 1989; Smith 1990; Huang & Khomami 2000), while in viscoelastic flows the elastic destabilization (i.e. in the limit of zero Reynolds number) is due to the coupling of base flow and perturbation velocities and stresses as well as their gradients at the free surface (Huang & Khomami 2000).

Unlike multilayer plane Poiseuille flows of viscoelastic fluids (Ganpule & Khomami 1998, 1999*a*, *b*) the interfacial and free-surface instability of viscoelastic fluids is not a very sensitive function of the thickness of the individual layers. Specifically, for Newtonian and viscoelastic flows in the limit of long waves, the interfacial instability is independent of individual layer thicknesses (Huang & Khomami 2000). Hence, one cannot effectively control interfacial and free-surface instabilities in flow down inclined planes by variation of the layer thicknesses, as is commonly done in multilayer plane Poiseuille flows of viscoelastic liquids (Ganpule & Khomami 1998, 1999*a*, *b*). Moreover, in multilayer flows of viscoelastic fluids it is very difficult to simultaneously obtain a stable free surface and an interface (Huang & Khomami 2000); hence, it is imperative that other strategies for control of free-surface and interfacial instabilities be explored.

There are many examples where the addition of an oscillatory component to a steady flow has been shown to reduce or enhance the stability of the flow. In fact, a number of studies have shown that time-periodic modulation of the basic flow can significantly influence the stability of bulk and free-surface flows. Grosch & Salwen (1968) and von Kerzek (1982) have studied the effect of pressure drop modulations on the linear stability of Newtonian plane Poiseuille flow and have demonstrated that in a certain frequency and amplitude regime dynamic modulation can be used to stabilize this flow. However, at very high or low frequencies the modulated flow is less stable. In fact, these studies have demonstrated that pressure modulations can alter the generation of shear waves in plane Poiseuille flow; hence, the frequency of most effective modulation is of the same order of magnitude as the most dangerous linearly unstable mode.

The effect of dynamic modulation on the stability of Newtonian flow down an inclined plane has also been investigated. Yih (1968) considered the effect of modulation of plate velocity on free-surface stability of a liquid film on a horizontal plate and demonstrated that in a specific range of amplitudes and frequencies the flow can become unstable due to the synchronous surface waves generated by oscillation of the plate. Recently, Lin, Chen & Woods (1996) as well as Lin & Chen (1998) have studied a similar problem but for a vertical plate and have shown that this inherently unstable vertical film flow can be stabilized by modulation of the plate with a certain frequency and amplitude. Although these studies have considered the effect of dynamic modulation on single-layer Newtonian flows, the effect of modulation frequency and amplitude on the stability of inclined plane flows of arbitrary inclination angles have not been considered. Moreover, the dependence of the most effective modulation frequencies on *Re* has not been investigated.

The studies on the effect of dynamic modulation of viscoelastic flows have been limited to bulk flows. Specifically, Ramanan, Kumar & Graham (1999) and Ramanan & Graham (2000) have considered the modulated viscoelastic Taylor–Couette flow. In their studies the effect of axial and angular modulations of the inner cylinder on the linear stability of this flow in the limit of small gap and vanishing Reynolds number was examined. Their results demonstrate that the extent of stabilization achieved by parallel superposition (i.e. angular modulation) is significantly smaller than that by the axial modulation. In addition, they demonstrated that for parallel superposition modulation frequencies near the inverse of the mean relaxation time of the fluid as well as frequencies near zero lead to significant stabilization of the flow while for axial modulation these frequencies lead to significant destabilization.

Clearly the studies mentioned above have demonstrated that dynamic modulation can be effectively used to influence the stability to bulk and free-surface flows. In addition, it has been shown that by combining the effect of fluid elasticity and dynamic modulation one can influence the stability of bulk viscoelastic flows. However, the influence of dynamic modulation on free-surface and interfacial stability of viscoelastic fluids is not well understood. In this study the effect of dynamic modulation on the stability characteristics of flow down an inclined plane is examined. The paper is organized as follows. The problem formulation and the method of solution are

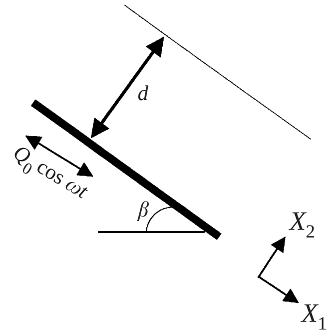

FIGURE 1. Schematic of the single-layer flow down an inclined plane in the presence of dynamic modulation.

summarized in § 2. In § 3 the results of our analyses are presented. Finally, the conclusions are presented in § 4.

2. Problem formulation

Figure 1 depicts the flow geometry considered in this study. We consider single-layer flow of Newtonian and viscoelastic fluids down an inclined plane.

2.1. *Governing equations*

The equations of motion and continuity can be expressed as

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla P - \nabla \cdot \boldsymbol{\tau} + \rho \boldsymbol{g}, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2}$$

where $\boldsymbol{u}, P, \boldsymbol{g}$, and $\boldsymbol{\tau}$ denote the velocity vector, isotropic pressure, gravitational acceleration and the deviatoric stress tensor. Our aim in this study is to identify the effect of dynamic modulation on the stability of viscoelastic flows down an inclined plane; hence, we have selected a relatively simple constitutive equation for the polymeric stress (i.e. the upper-convected Maxwell (UCM) model). The UCM model can be derived from a molecular theory in which the polymer molecules are modelled as non-interacting Hookean elastic dumbbells (Bird *et al.* 1987). Although this model gives rise to relatively simple material properties (i.e. constant viscosity and first normal stress coefficient), it retains the essential physics necessary for investigating the effects of fluid elasticity on stability of this class of flows.

The UCM model is given by

$$\boldsymbol{\tau}_p + \lambda \boldsymbol{\tau}_{p(1)} = -\eta_p \dot{\boldsymbol{\gamma}}, \tag{3}$$

where η_p and λ are the polymer viscosity and relaxation time,

$$\boldsymbol{X}_{(1)} = \frac{\partial \boldsymbol{X}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{X} - (\nabla \boldsymbol{U})^T \cdot \boldsymbol{X} - \boldsymbol{X} \cdot \nabla \boldsymbol{U}, \tag{4}$$

and the deformation rate tensor is given by

$$\dot{\boldsymbol{\gamma}} = \nabla \boldsymbol{U} + (\nabla \boldsymbol{U})^T. \tag{5}$$

The following set of dimensionless variables is introduced to non-dimensionalize the above equations:

$$\left.\begin{aligned} x &= \frac{x_1}{d_1}, \quad y = \frac{x_2}{d_1}, \quad Re = \frac{\rho g \sin\beta\, \mathrm{d}^2}{\eta}, \\ \mathrm{P} &= \frac{\bar{\mathrm{P}}}{\eta\, U_0/d_1}, \quad \tau_{ij} = \frac{\bar{\tau}_{ij}}{\eta\, U_0/d_1}, \quad S = \frac{\sigma}{\eta\, U_0}, \\ We &= \frac{\lambda U_0}{d}, \quad t = \frac{t^* U_0}{d}, \quad \Omega = \frac{\omega d}{U_0}, \quad \Lambda = \frac{Q_0}{U_0}, \end{aligned}\right\} \tag{6}$$

where U_0 is the velocity at the free surface, d is the thickness of the fluid layer, ρ is the fluid density, σ is the surface tension, β is the inclination angle, and ω and Q_0 are the frequency and amplitude of the modulation velocity respectively.

2.2. *Base flow*

In the presence of dynamic modulation the base flow is time dependent. Hence, to simplify the analysis, the base flow velocity and stress fields have been split into a steady and a transient contribution,

$$\left.\begin{aligned} \boldsymbol{u}(y,t) &= \boldsymbol{u}_{ss}(y) + \boldsymbol{u}_D(y,t), \\ \boldsymbol{\tau} &= \boldsymbol{\tau}_{ss}(y) + \boldsymbol{\tau}_D(y,t), \\ \mathrm{p} &= \mathrm{p}_{ss}(y) + p_D(y,t). \end{aligned}\right\} \tag{7}$$

Upon substitution of the above equation into the governing equations, the following set of equations is obtained:

$$\left(\frac{\partial u_{ss}}{\partial x} + \frac{\partial v_{ss}}{\partial y}\right) + \left(\frac{\partial u_D}{\partial x} + \frac{\partial v_D}{\partial y}\right) = 0, \tag{8}$$

$$Re\frac{\partial u_D}{\partial t} = \left[-\left(\frac{\partial P}{\partial x}\right)_{ss} - \left(\frac{\partial \tau_{xy}}{\partial y}\right)_{ss} + \rho\, g_x\right] + \left[-\left(\frac{\partial P}{\partial x}\right)_D - \left(\frac{\partial \tau_{xy}}{\partial y}\right)_D\right], \tag{9}$$

$$We\left(\frac{\partial \tau_{xx}}{\partial t}\right)_D = \left[-(\tau_{xx})_{ss} + 2(\tau_{xy})_{ss}\, We\frac{\partial u_{ss}}{\partial y}\right] + \left[-(\tau_{xx})_D + 2(\tau_{xy})_D\, We\frac{\partial u_D}{\partial y}\right], \tag{10}$$

$$We\left(\frac{\partial \tau_{xy}}{\partial t}\right)_D = \left[-(\tau_{xy})_{ss} - \left(\frac{\partial u_{ss}}{\partial y}\right)\right] + \left[-(\tau_{xy})_D - \left(\frac{\partial u_D}{\partial y}\right)\right]. \tag{11}$$

The boundary conditions are given by

$$u_D(y=0,t) = \Lambda\cos(\Omega t); \quad u_{ss}(y=0) = 0, \tag{12}$$

and vanishing of shear stress at the free surface yields the equation

$$\tau_{xy,ss}(y=d) = \tau_{xy,D}(y=d,t) = 0. \tag{13}$$

The steady part of the base flow solution can be obtained analytically:

$$\left.\begin{aligned} u_{ss} &= 2y - y^2, \\ \tau_{xx,ss} &= -8We(1-y)^2; \tau_{yx,ss} = 2(\mathrm{y}-1), \\ \tau_{yy,ss} &= 0. \end{aligned}\right\} \tag{14}$$

To solve the time-dependent part of the base flow the dependent variables have been discretized as follows:

$$u_D(t,y) = \sum_{i=1}^{N+1} a_i(t)T_i(y), \quad \tau_{ij,D}(t,y) = \sum_{i=1}^{N+1} b_i(t)T_i(y). \tag{15}$$

The spatial discretization is performed by using the Chebyshev-tau method described in detail in our earlier publications (Ganpule & Khomami 1998, 1999*a*, *b*; Huang & Khomami 2000). For the equation of motion, $\bar{N} = N - 2$ (i.e. $\bar{N}$ is the number of coefficients retained), and for each component of the constitutive equation, $\bar{N} = N$. For the UCM model this method of discretization yields a $(4N + 4)$ by $(4N + 4)$ coefficient matrix. Temporal discretization has been performed using the finite difference technique, specifically a first-order forward difference method.

2.3. *Perturbation flow*

For the constitutive equation used in this study Squire's theorem is valid. However, the extension of Squire's theorem (Talpa & Bernstein 1970) to time-dependent flows has not been demonstrated. As a result, we have performed a few calculations with three-dimensional disturbances. Our limited computations demonstrated the validity of this theorem. However, a much more extensive study is required in order to prove the extension of Squire's theorem to time-dependent viscoelastic flows. Considering the tremendous computational effort required to perform three-dimensional analyses, we have assumed (based on our limited computations) that the extension of Squire's theorem to time-dependent viscoelastic flows is valid. Hence, we have only considered the stability of the flow to two-dimensional disturbances. The procedure for performing linear stability analysis in the presence of dynamic modulation is similar to that for steady base-state flows with the exception that the perturbation vector $\boldsymbol{Z}^p$ is defined as follows:

$$\boldsymbol{Z}^p = [\mathrm{u}'_\mathrm{D}, \mathrm{v}'_\mathrm{D}, \mathrm{p}'_\mathrm{D}, \tau'_{\mathrm{ij,D}}] = [\mathrm{U_D(y,t), V_D(y,t), f_D(y,t), F_{ij,D}(y,t)}] \exp(\mathrm{i}\alpha \mathrm{x}). \tag{16}$$

The linear stability equations are derived by substituting equation (16) into the governing equations and keeping only the terms that are linear with respect to the perturbation quantities. To reduce the number of variables, the linear stability equations are recast in terms of the perturbation stream function Ψ defined as

$$\Psi = \phi(\mathrm{y,t}) \exp(\mathrm{i}\alpha x), \tag{17}$$

where

$$u' = \frac{\partial \Psi}{\partial y} \quad \text{and} \quad v' = -\frac{\partial \Psi}{\partial x}. \tag{18}$$

In turn the pressure terms in the x- and y-components of the linearized equation of motion are eliminated by cross-differentiation and the stability governing equations are obtained:

$$Re\frac{\partial}{\partial t}(\mathrm{L}\phi) = -\mathrm{i}\alpha Re\left(u\mathrm{L}\phi - \frac{\partial^2 u}{\partial y^2}\right) + \mathrm{i}\alpha\left(\frac{\partial F_{yy}}{\partial y} - \frac{\partial F_{xx}}{\partial y}\right) - \left(\alpha^2 + \frac{\partial^2}{\partial y^2}\right)F_{xy}, \tag{19}$$

$$We\frac{\partial F_{xx}}{\partial t} = -F_{xx} - We\left\{\mathrm{i}\alpha U F_{xx} - \mathrm{i}\alpha\frac{\partial \tau_{xx}}{\partial y}\phi - 2\left[\mathrm{i}\alpha\tau_{xx}\frac{\partial\varphi}{\partial y} + \tau_{xy}\frac{\partial^2\varphi}{\partial y^2} + \frac{\partial U}{\partial y}F_{xy}\right]\right\} -2\mathrm{i}\alpha\frac{\partial\varphi}{\partial y}, \tag{20}$$

$$We\frac{\partial F_{xy}}{\partial t} = -F_{xy} - We\left\{\mathrm{i}\alpha U F_{xy} - \mathrm{i}\alpha\frac{\partial \tau_{xy}}{\partial y}\phi - \alpha^2\tau_{xx}\phi - \frac{\partial U}{\partial y}F_{yy}\right\} - \left(\frac{\partial^2\phi}{\partial y^2} + \alpha^2\phi\right), \tag{21}$$

$$We\frac{\partial F_{yy}}{\partial t} = -F_{xy} + 2We\ \alpha^2\tau_{xy}\phi - \mathrm{i}\alpha U\ We\ F_{yy} + 2\mathrm{i}\alpha\frac{\partial\phi}{\partial y}, \tag{22}$$

where

$$\mathrm{L} = (\partial^2/\partial y^2 - \alpha^2).$$

The boundary conditions for the perturbation flow are as follows:
no slip at the solid boundary

$$\phi = 0 \quad \text{and} \quad \frac{\mathrm{d}\phi}{\mathrm{d}y} = 0; \tag{23}$$

and balance of tangential and normal stresses at the free surface (i.e. at $y = 1$):

$$F_{xy} + \frac{\mathrm{d}\tau_{xy}}{\mathrm{d}y}h_s - \mathrm{i}\alpha(\tau_{xx} - \tau_{yy})h_s = 0, \tag{24}$$

$$Re\frac{\partial^2\phi}{\partial t\partial y} = \mathrm{i}\alpha Re\left(\phi\frac{\mathrm{d}U}{\mathrm{d}y} - U\frac{\mathrm{d}\phi}{\mathrm{d}y}\right) - \mathrm{i}\alpha(F_{xx} - F_{yy}) - \frac{\partial F_{xy}}{\partial y} - \mathrm{i}\alpha(\cot\beta + \alpha^2 S)h_s. \tag{25}$$

In the above equations, the free-surface position can be expressed as

$$\frac{\partial h_s}{\partial t} = -\mathrm{i}\alpha\phi - \mathrm{i}\alpha U\, h_s,$$

where h_s denotes the amplitude of free-surface deviation from its unperturbed position.

2.4. *Solution procedure*

The dominant mode of the instability can be due to disturbances of arbitrary wavelength. Hence, the eigenvalue problem constituted by the stability governing equations must be solved numerically. The numerical method used is once again the multi-domain spectral-tau method that we have shown to provide an accurate solution to this class of problems (Ganpule & Khomami 1998, 1999*a,b*; Huang & Khomami 2000). Specifically, the dependent variables are expanded in terms of Chebyshev polynomials with time-dependent coefficients, $a_i(t)$:

$$\left.\begin{aligned} \phi(t,y) &= \sum_{i=1}^{N+1} a_i(t)T_i(y); \quad F_{xy}(t,y) = \sum_{i=1}^{N+1} a_{3(N+1)+i}(t)T_i(y), \\ F_{xx}(t,y) &= \sum_{i=1}^{N+1} a_{N+1+i}(t)T_i(y); \quad F_{yy}(t,y) = \sum_{i=1}^{N+1} a_{2(N+1)+i}(t)T_i(y). \end{aligned}\right\} \tag{26}$$

After substituting the above quantities into the governing equations and boundary conditions (i.e. equations (19)–(25)), one obtains the stability governing equations that can be written as

$$\boldsymbol{Q}\frac{\mathrm{d}\boldsymbol{a}(t)}{\mathrm{d}t} = \boldsymbol{P}\boldsymbol{a}(t) \tag{27}$$

where $\boldsymbol{Q}$ and $\boldsymbol{P}$ are coefficient matrices (i.e. $[4(N+1)+1]$ by $[4(N+1)+1]$), and $\boldsymbol{a}$ is a vector that contains the time-dependent expansion coefficients. Equation (27) constitutes a time-dependent eigenvalue problem. Since at low modulation frequencies it is very difficult to get accurate results with first-order time integration schemes, we have selected the fourth-order Runge–Kutta method to perform the time integration. It should be noted that because of the nature of boundary conditions (i.e. not all variables are represented in all the boundary conditions), all the entries of some of the rows of the $\mathbf{Q}$ matrix are zero (i.e. this matrix is singular). To remedy this situation, we have chosen to explicitly rewrite the boundary conditions in terms of the expansion coefficients and by a process of elimination remove the rows that contain only zeros. This procedure is illustrated below.

(i) No slip at the solid wall:

$$\frac{\mathrm{d}\phi}{\mathrm{d}y} = 0, \rightarrow \sum_{i=1}^{N+1} a_i \frac{\partial T_i(\bar{y} = -1)}{\partial y} = 0. \tag{28}$$

$$\phi = 0, \rightarrow \sum_{i=1}^{N+1} a_i T_i(\bar{y} = -1) = 0. \tag{29}$$

Hence, combining equations (28) and (29) a_1 and a_2 can be written in terms of the other coefficients as follows:

$$a_1 = \sum_{i=3}^{N+1} \left[\frac{T_2(\bar{y} = -1)}{\partial T_2(\bar{y} = -1)/\partial y} \frac{\partial T_i(\bar{y} = -1)}{\partial y} - T_i(\bar{y} = -1) \right] a_i, \tag{30}$$

$$a_2 = -\frac{1}{\partial T_2(\bar{y} = -1)/\partial y} \sum_{i=3}^{N+1} \frac{\partial T_i(\bar{y} = -1)}{\partial y} a_i. \tag{31}$$

(ii) The tangential stress balance at the free surface ($y = 1$)

$$F_{xy} + \frac{\mathrm{d}\bar{\tau}_{xy}}{\mathrm{d}y} \bar{h}_s - \mathrm{i}\alpha(\bar{\tau}_{xx} - \bar{\tau}_{yy})\bar{h}_s = 0. \tag{32}$$

Therefore,

$$\sum_{i=1}^{N+1} a_{3(N+1)+i} T_i(\bar{y} = 1) + \frac{\mathrm{d}\bar{\tau}_{xy}}{\mathrm{d}y} \bar{h}_s - \mathrm{i}\alpha(\bar{\tau}_{xx} - \bar{\tau}_{yy})\bar{h}_s = 0. \tag{33}$$

Hence, utilizing equation (33) $a_3(N+1)+1$ can be written in terms of the other expansion coefficients as follows:

$$a_{3(N+1)+1} = -\sum_{i=2}^{N+1} a_{3(N+1)+i} T_i(\bar{y} = 1) - \left[\frac{\mathrm{d}\bar{\tau}_{xy}}{\mathrm{d}y} \bar{h}_s - \mathrm{i}\alpha(\bar{\tau}_{xx} - \bar{\tau}_{yy})\bar{h}_s \right]. \tag{34}$$

Following this procedure equation (27) can be written as

$$\mathbf{Q}' \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \mathbf{P}' \boldsymbol{a} \tag{35}$$

where $\mathbf{Q}'$ and $\mathbf{P}'$ are $[4(N+1)+1-3]$ by $[4(N+1)+1-3]$ non-singular matrices. Hence, matrix $\mathbf{Q}'$ can be inverted (the matrix inversion has been performed using subroutines ZGETRF and ZGETRI from the public website netlib), and equation (35) can be written as

$$\mathbf{Q}' \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \mathbf{P}' \boldsymbol{a} \rightarrow \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \mathbf{A}(t)\boldsymbol{a}, \tag{36}$$

where $\boldsymbol{a} = (a_1, a_2, \ldots, a_{4(N+1)}, \bar{h}_s)$, $\mathbf{A}(t) = (\mathbf{Q}'^{-1}\mathbf{P}')$.

Equation (36) can be easily integrated forward in time. Since $\boldsymbol{A}(t)$ is a time-periodic matrix, the disturbance amplitude could grow or decay during one cycle of modulation. Therefore, one needs to develop a consistent criterion for determining the effect of dynamic modulation on the stability of the flow over one modulation cycle. To accomplish this, we have used Floquet theory that allows one to determine whether a disturbance on average grows or decays over one modulation cycle. Specifically, since the modulations considered in this study are time periodic, Floquet theory clearly shows that there must exist a constant matrix $\boldsymbol{F}$, such that for all t (Ioos & Joseph 1990)

$$\boldsymbol{X}(t+T) = \boldsymbol{F}\boldsymbol{X}(T), \tag{37}$$

where T is the modulation period, and $\boldsymbol{X}$ is the fundamental solution matrix satisfying

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{A}(t)\boldsymbol{X}. \tag{38}$$

In turn, the Floquet exponents which determine the average growth or decay of a disturbance over a cycle can be determined as follows:

$$\lambda_i = \frac{1}{T}\rho_i, \tag{39}$$

where ρ_i are the eigenvalues of the $\boldsymbol{F}$ matrix (i.e. the QR algorithm is used to determine these eigenvalues), and λ_i are the Floquet exponents. In turn, the stability of the flow is determined by examining λ_i (i.e. the flow is linearly stable if the real part of λ_i is negative; if the real part of λ_i is positive, the flow is unstable. Otherwise, it is neutrally stable).

To apply Floquet theory to the problem at hand, one needs to determine matrix $\boldsymbol{F}$. To determine this matrix, equation (36) is integrated over a period of modulation. It should be noted that,

$$\boldsymbol{X}(T) = \boldsymbol{F}\boldsymbol{X}(0), \tag{40}$$

where $\boldsymbol{X}(0)$ can be chosen arbitrarily without loss of generality; hence we have chosen $\boldsymbol{X}(0) = \boldsymbol{I}$, where $\boldsymbol{I}$ is the unit tensor. To ensure the accuracy of our hybrid time-integration/eigenvalue scheme we have performed extended validation studies with various $\boldsymbol{A}$ matrices. In all cases, accurate results were obtained by both the fourth-order Runge–Kutta (RK4) method as well as the first-order Euler method (see table 1 for representative results). However, in all cases it was observed that the first-order Euler method requires tremendously small time increments to provide an accurate solution. Hence throughout this study the RK4 method has been used in the analyses.

Clearly the above procedure (i.e. time integration to generate matrix $\boldsymbol{F}$ combined with determination of all the eigenvalues of matrix $\boldsymbol{F}$) can also be used to determine the stability of flows in the absence of dynamic modulation. Specifically, this is accomplished by integration of equation (36) forward in time and evaluation of the eigenvalues of the coefficient matrix at long times. To examine the accuracy of our hybrid numerical technique, we have examined the linear stability of viscoelastic flows down an inclined plane in the absence of dynamic modulation using both the combined time-integration eigenvalue analysis strategy and classical eigenvalue analysis. The results obtained from the generalized eigenvalue problem (GEVP) (the GEVP results were obtained using the same procedure as our earlier studies (Ganpule & Khomami 1998, 1999*a, b*; Huang & Khomami 2000)) and the hybrid scheme have been compared. Table 2 shows a comparison of free surface-eigenvalues obtained

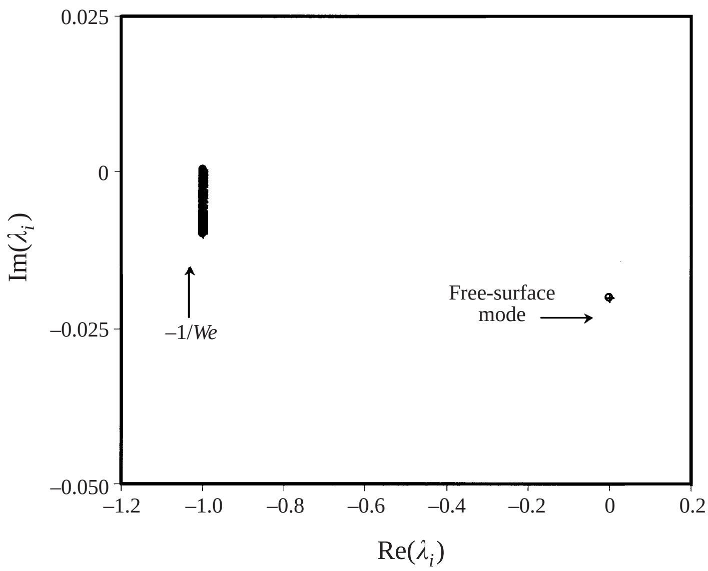

FIGURE 2. The spectrum of eigenvalues for one-layer UCM fluid down an inclined plane, $Re = 0.01$, $\beta = 11.5°$, $We = 1$, $\alpha = 0.01$, $S = 0$: +, spectrum obtained based on the GEVP; ○, spectrum obtained by using the hybrid time-integration eigenvalue technique.

Δt (time step)	Ω (frequency)	τ (period)	Analytical results	Computational results
$\pi/10$	1	2π	$\lambda = 0$	$\lambda = -2.68 \times 10^{-4}$
			$\lambda = 1$	$\lambda = 0.999893194$
$\pi/50$	1	2π	$\lambda = 0$	$\lambda = -6.30 \times 10^{-8}$
			$\lambda = 1$	$\lambda = 0.999999783$
$\pi/100$	1	2π	$\lambda = 0$	$\lambda = -1.98 \times 10^{-9}$
			$\lambda = 1$	$\lambda = 0.999999986$

TABLE 1. A comparison between results obtained analytically and those of the fourth-order Runge–Kutta (RK4) method.

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}T} = \boldsymbol{AX} \quad \text{where } \boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & h(t) \end{bmatrix} \quad \text{and} \quad h(t) = \frac{[\cos(t) + \sin(t)]}{[2 + \sin(t) - \cos(t)]}; \quad \lambda = \frac{1}{\tau}\ln(\rho).$$

The ρ are the eigenvalues of $\boldsymbol{A}$; the λ are the Floquet exponents.

by the hybrid time-integration/eigenvalues analysis scheme to the generalized eigenvalue analysis. The excellent agreement between these results clearly demonstrates the accuracy of the hybrid scheme used in this study. A more stringent test of the accuracy of the combined time-integration/eigenvalue scheme would involve comparing the spectrum of eigenvalues obtained by both techniques. Figure 2 shows such a comparison: clearly the spectrum obtained based on the combined time-dependent integration/eigenvalue method is identical to the spectrum of eigenvalues obtained from a generalized eigenvalue analysis. Moreover, it should be noted that the free-surface mode is well separated from the continuous spectrum (i.e. the continuous spectrum is located at $\mathrm{Im}(\alpha c) \sim -1/We$ (Huang & Khomami 2000)). This free-

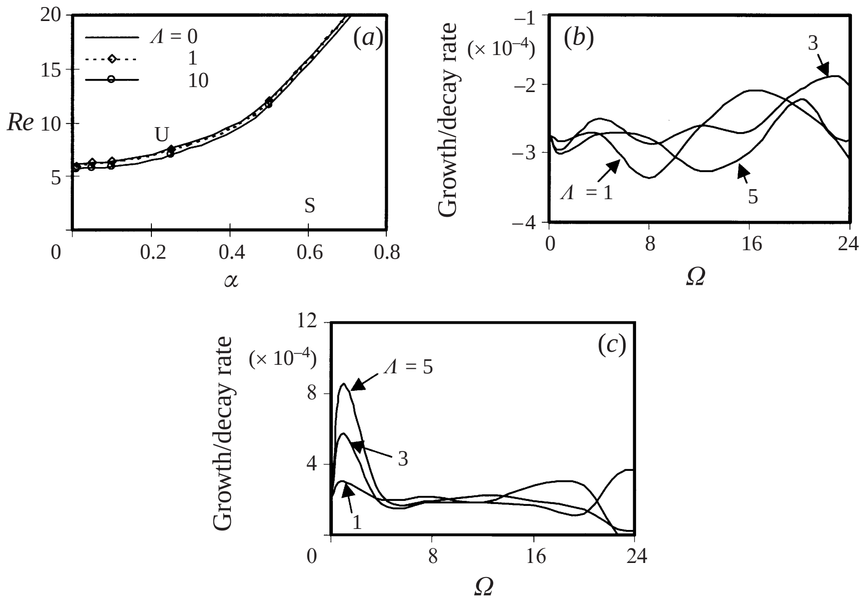

FIGURE 3. The effect of dynamic modulation on the free-surface instability of one-layer Newtonian flow down an inclined plane, $\beta = 11.5°$, $S = 0$. (*a*) Neutral stability diagram $\Omega = 1$; S denotes stable, U denotes unstable; (*b*) growth/decay rate, $\alpha = 0.01$, $Re = 1.0$; (*c*) growth/decay rate, $\alpha = 0.1$, $Re = 10$.

β (deg.)	We	GEVP ($\lambda = -i\alpha c$)	Time Integration ($\Lambda = 0$, $\Omega = 1$, $T = 2\pi\Omega$)
11.5	0.5	$-2.61677 \times 10^{-4} - 2.0 \times 10^{-2}$ i	$-2.61584 \times 10^{-4} - 1.99972 \times 10^{-2}$ i
	1	$-1.95010 \times 10^{-4} - 2.0 \times 10^{-2}$ i	$-1.94928 \times 10^{-4} - 1.99948 \times 10^{-2}$ i
	2	$-6.16770 \times 10^{-5} - 2.0 \times 10^{-2}$ i	$-6.20004 \times 10^{-5} - 1.99852 \times 10^{-2}$ i
78.5	0.5	$5.36307 \times 10^{-5} - 2.0 \times 10^{-2}$ i	$5.35982 \times 10^{-5} - 1.99972 \times 10^{-2}$ i
	1	$1.20297 \times 10^{-4} - 2.0 \times 10^{-2}$ i	$1.20164 \times 10^{-4} - 1.99948 \times 10^{-2}$ i
	2	$2.5363 \times 10^{-4} - 2.0 \times 10^{-2}$ i	$2.52836 \times 10^{-4} - 1.99852 \times 10^{-2}$ i

TABLE 2. Comparison of free-surface eigenvalues obtained by time-integration (i.e. based on the Floquet exponents) and from the GEVP for one-layer UCM flow down an inclined plane. $Re = 0.01$, $\alpha = 0.01$, $S = 0$. λ is the Floquet exponent, c is the eigenvalue obtained from the GEVP. Λ is the amplitude; Ω is the frequency and T is the period of the modulation.

surface mode is the only mode that will be discussed in the remainder of this paper.

3. Results and discussion

The stability of one-layer flow down an inclined plane is a function of the inclination angle, Re, S, We, and the disturbance wavenumbers. To examine the effect of dynamic modulation on the stability characteristics of this class of flows, we have performed a comprehensive linear stability analysis of Newtonian and viscoelastic flow down an inclined plane. Of particular interest is the relationship between the modulation frequency and Re and We.

3.1. *Newtonian flows*

Utilizing the above-mentioned numerical technique, the linear stability of dynamically modulated one-layer Newtonian flow down an inclined plane has been examined. Since the parameter space is relatively large, we have concentrated on the long-wave regime since the most dangerous disturbances for this class of flows are long wave. Figure 3(*a*) shows the stability contour in the presence of dynamic modulation. Clearly, at low to moderate *Re* the effect of dynamic modulation on the stability characteristic of one-layer Newtonian flows is not very significant. Figures 3(*b*) and 3(*c*) depict the influence of inertia on the role of dynamic modulation. Clearly, as the Reynolds numbers increased, the influence of dynamic modulation on the stability of the flow is enhanced. This can be rationalized by the fact that in the limit of zero Reynolds number, dynamic modulation cannot influence the stability of Newtonian flows down an inclined plane since the problem is invariant under arbitrary rigid motions of the reference frame (i.e. the problem is Euclidean invariant). At low *Re* where the influence of dynamic modulation is insignificant the most dominant modulation frequency does not scale with the inertial time scale (i.e. $1/Re$). It should be noted that these conclusions are consistent with the results of Lin & Chen (1998). However, at higher Reynolds numbers, where dynamic modulation destabilizes the flow, the frequency of modulation corresponding to the maximum destabilization occurs in the vicinity of the inertial time scale (i.e. $1/Re$). Moreover, the extent of destabilization scales with the amplitude of modulation. It should be noted that these trends are independent of the inclination angle.

Overall, for reasons mentioned above, parallel superposition modulations (i.e. modulation of the plane surface parallel to the mean flow) cannot effectively be used to stabilize or destabilize free-surface disturbance in one-layer Newtonian flows down an inclined plane at low *Re*. Although, the effect of parallel superposition modulations are more pronounced at moderate *Re*, significant destabilization is only observed for very large modulation amplitudes.

3.2. *Viscoelastic flows*

Figure 4 shows the stability contour in the presence of dynamic modulation for one-layer UCM flow down an inclined plane. Clearly, the effect of dynamic modulation on the stability characteristics of viscoelastic flows is much more pronounced than on those of Newtonian flows. Hence, we have chosen to systematically investigate the stability of one-layer UCM flows down an inclined plane in the presence of parallel superposition of modulation.

Figures 4(*b*) and 4(*c*) demonstrate the effect of dynamic modulation on the free-surface instability at various amplitudes and frequencies of modulation in the limit of vanishing *Re*. At small inclination angles (i.e. $\beta = 11.5°$) and in the limit of long waves, it is observed that higher modulation frequencies tend to destabilize free-surface disturbances while lower frequencies have the opposite effect. Moreover, the extent of stabilization or destabilization is proportional to the amplitude of the modulation (see figure 4*b*). This trend also holds in the limit of $O(1)$ waves (see figure 4*c*). Figure 5 shows the effect of dynamic modulation at a higher inclination angle ($\beta = 78.5°$). Clearly, increasing the inclination angle does not change the general stabilizing or destabilizing effects of dynamic modulation.

To examine the dependence of the most effective modulation frequencies on the elasticity of the fluid, we have examined a wide range of flows with different *We*. A representative set of results in the limit of vanishing *Re* is shown in figure 6. As shown by this figure the dominant stabilizing frequency is reduced as the *We* is increased. Our

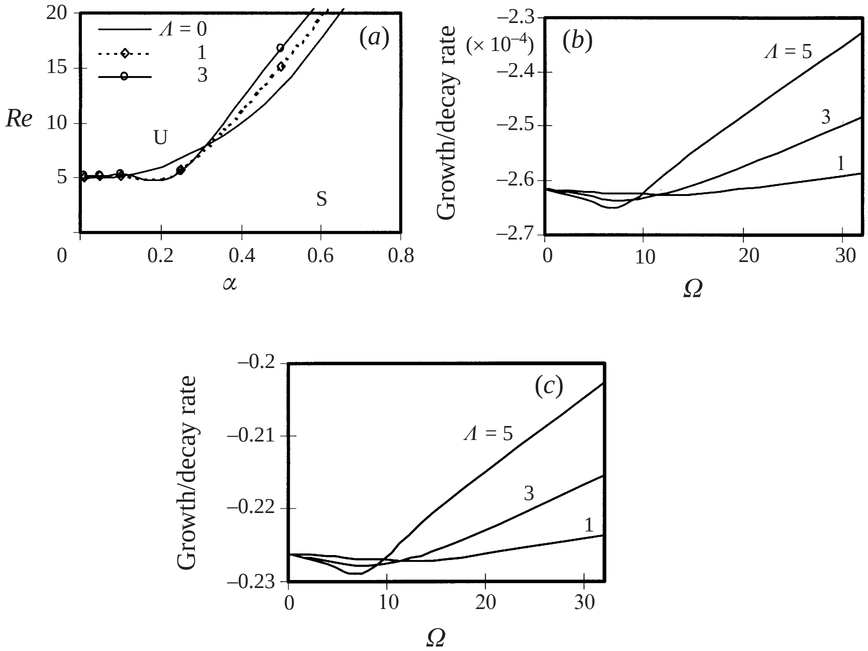

FIGURE 4. The effect of dynamic modulation on the free surface instability of one-layer UCM flow down an inclined plane, $\beta = 11.5°$, $We = 0.5$, $S = 0$. (*a*) Neutral stability diagram; S denotes stable, U denotes unstable; (*b*) growth/decay rate, $Re = 0.01$, $\alpha = 0.01$; (*c*) growth/decay rate, $Re = 0.01$, $\alpha = 1$.

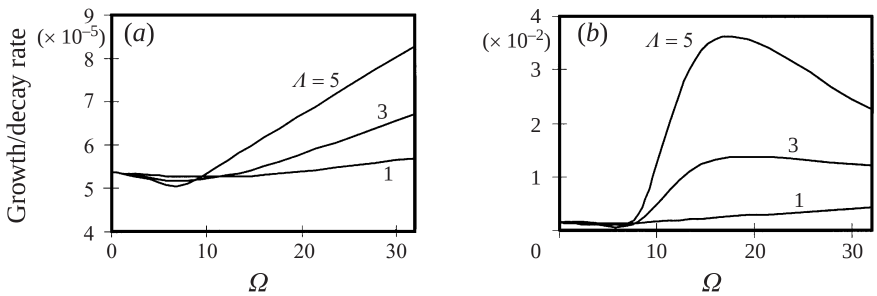

FIGURE 5. The effect of dynamic modulation on the free surface-instability of one-layer UCM flow down an inclined plane, $\beta = 78.5°$, $Re = 0.01$, $We = 0.5$, $S = 0$. (*a*) $\alpha = 0.01$; (*b*) $\alpha = 2$.

computations also show that this trend is independent of inclination angle and the stabilization/destabilization is relatively insensitive to the disturbance wavenumber (i.e. in the long and $O(1)$ wave limit). Moreover, the extent of stabilization is shown to be proportional to the amplitude of modulation.

Inertia also plays a significant part in determining the role of dynamic modulation on the stability of viscoelastic flows down an inclined plane. In fact, at $O(1)$ Re, the role of dynamic modulation on the stability of one-layer UCM flow down an inclined plane is very different from that at the low-Re limit. Figure 7 shows that for $O(1)$ Re lower-frequency modulations are destabilizing while intermediate-frequency modulations have a stabilizing effect. It should be noted that the destabilizing or stabilizing effect of dynamic modulation is proportional to the amplitude of the

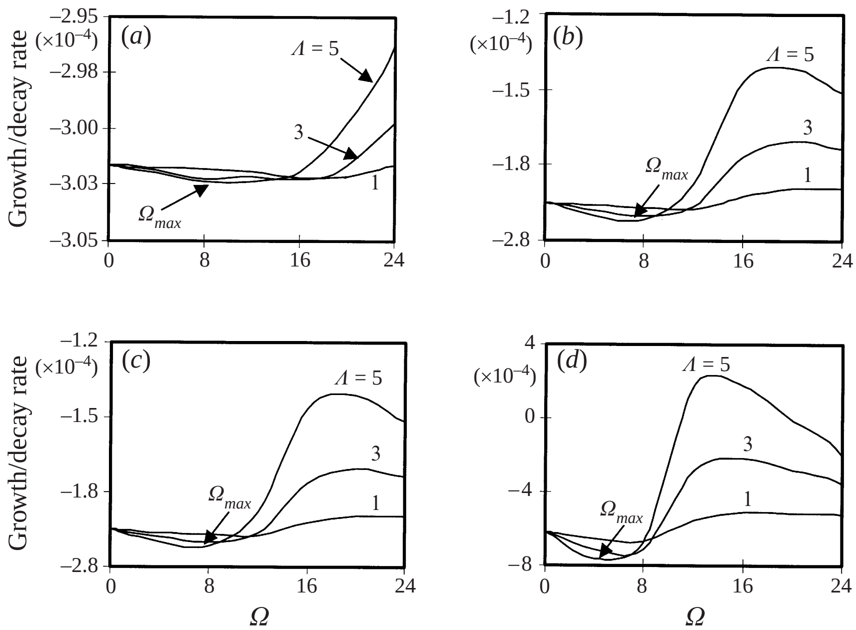

FIGURE 6. The effect of fluid elasticity on the free-surface instability of dynamically modulated one-layer UCM flow down an inclined plane. $\beta = 11.5°$, $Re = 0.01$, $\alpha = 0.01$, $S = 0$. (*a*) $We = 0.2$, (*b*) $We = 0.5$, (*c*) $We = 1.0$, (*d*) $We = 2.0$.

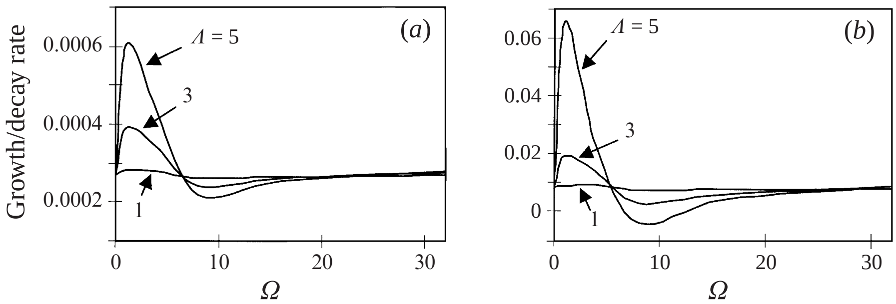

FIGURE 7. The effect of fluid inertia on the free-surface instability of dynamically modulated one-layer UCM flow down an inclined plane, $\beta = 11.5°$, $Re = 10$, $We = 0.5$, $S = 0$. (*a*) $\alpha = 0.01$; (*b*) $\alpha = 0.35$.

modulation. Moreover, similarly to the low-*Re* limit, changing the inclination angle does not alter the influence of dynamic modulation on the stability of the free surface. In fact, as *Re* is progressively increased the above trends remain the same, with the only difference that the destabilization occurs at lower and lower frequencies. Overall, these analyses clearly suggest that for $O(1)$ *Re*, the combination of dynamic modulation and inertia leads to destabilization at low modulation frequencies. In fact, this is very similar to what was observed in Newtonian flows.

As shown above, the influence of low-frequency dynamic modulation on the stability of viscoelastic flows down an inclined plane is very complex; specifically, the fact that low-frequency modulation stabilizes the flow at vanishing *Re* while destabilizing the flow at higher *Re* (i.e. in the long and $O(1)$ wave limit) clearly shows that modulation of the flow in this frequency range leads to a competition between elastic and inertial

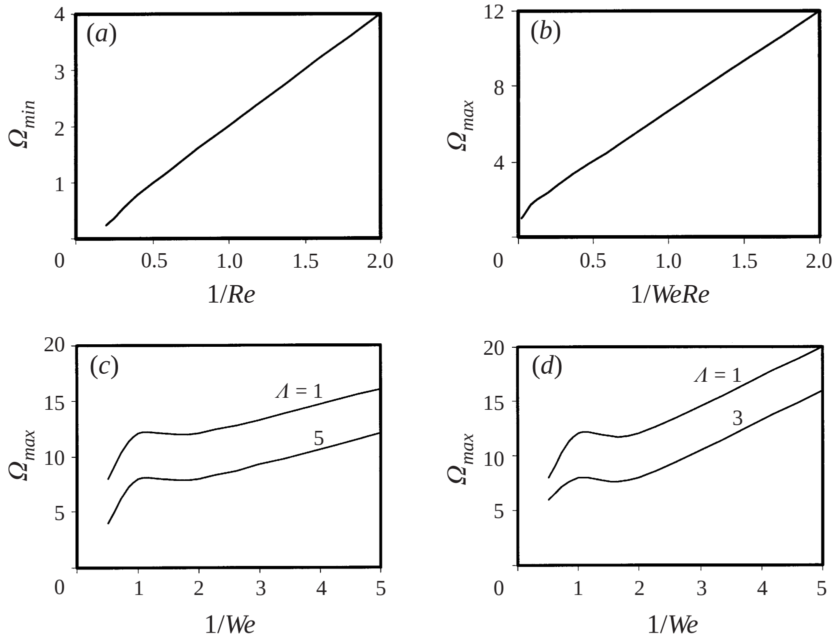

FIGURE 8. The most dominant modulation frequencies for one-layer UCM flow down an inclined plane. (*a*) $\beta = 11.5°$, $S = 0$, $E = We/Re = 1$ ($Re = 0.5, 2, 5$), most dominant destabilizing frequency scales with $1/Re$; (*b*) $\beta = 11.5°$, $S = 0$, $E = We/Re = 1$ ($Re = 0.5, 2, 5$), the most stabilizing frequency scale with $1/(WeRe)$; (*c*) $Re = 0.01$, $\beta = 11.5°$, $S = 0$; (*d*) $Re = 0.01$, $\beta = 78.5°$, $S = 0$.

forces. To better understand the relationship between the most dominant frequencies and Re and We, we have performed extensive computations. The results of these computations are summarized in figure 8. Figures 8(*a*) and 8(*b*) clearly show that for finite-Re flows the most dominant destabilizing frequency (i.e. Ω_{min}) scales with $1/Re$ while the most stabilizing frequencies scale with $1/ReWe$. In case of vanishing Re, the most stabilizing frequencies scale with $1/We$ as shown by figures 8(*c*) and 8(*d*). It should be noted that these trends are independent of the inclination angle.

To better understand the mechanism by which dynamic modulation influences the stability of one-layer viscoelastic flow down an inclined plane, the frequency of the free-surface wave as a function of modulation frequency has been examined. In the limit of long waves, increasing the modulation frequency will first decrease the frequency of the perturbed interface and then enhance it (see figure 9). This suggests that stabilization attained by dynamic modulation is a consequence of reducing the frequency of the free-surface wave. In fact, if the most dominant frequency (in terms of reducing the frequency of the free-surface wave) is determined, one observes that this frequency also scales with $1/We$ at vanishing Re irrespective of the inclination angle.

As mentioned throughout this section, dynamic modulation could have a stabilizing or destabilizing effect on the free-surface stability of one-layer UCM flows down an inclined plane. However, from a practical point of view one needs to examine the range of amplitudes and frequencies that can be used to stabilize the free-surface instability under particular operating conditions. To illustrate this point, we have selected a case where the flow is unstable in the limit of long waves (see figure 10). Clearly different combinations of dynamic modulation amplitudes and frequencies can be used to stabilize the free surface. Indeed, frequencies of $O(1/We)$ can be used to stabilize

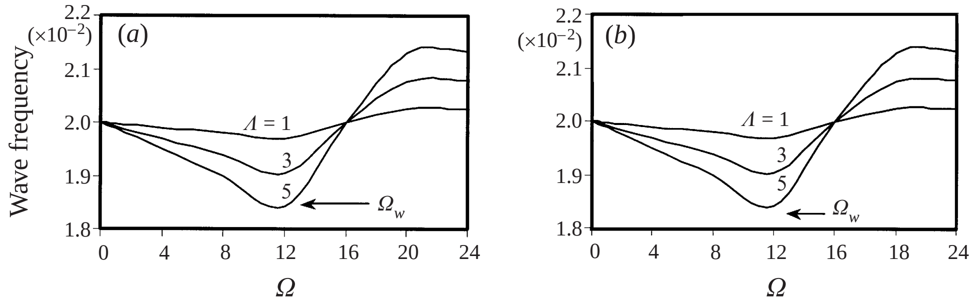

FIGURE 9. The free-surface wave frequency as a function of the modulation frequency for one-layer UCM flow down an inclined plane. $\alpha = 0.01$, $Re = 0.01$, $\beta = 11.5°$, $S = 0$. (*a*) $We = 0.5$; (*b*) $We = 1.0$. Ω_w is the most stabilizing modulation frequency.

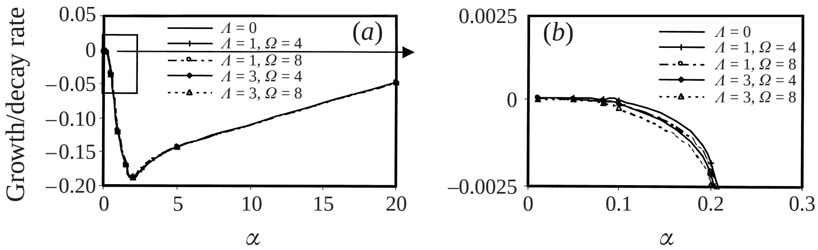

FIGURE 10. The influence of parallel superposition of modulation on the free-surface instability of one-layer flow of a UCM fluid down an inclined plane. $\beta = 45.3°$, $Re = 0.01$, $We = 0.5$, $S = 0$. (*a*) growth rate, (*b*) magnification of the marked area in (*a*).

the flow and the stabilizing influence of dynamic modulation is proportional to the amplitude of the modulation. However, relatively large modulation amplitudes have to be used to achieve significant stabilization with the parallel superposition modulation technique. This might limit the usefulness of this strategy in industrial applications.

3.3. *Stability mechanism*

The mechanism of instability of one-layer Newtonian flows down an inclined plane has been previously examined by Kelly *et al.* (1989) based on a rigorous energy analysis and by Smith (1990) via examination of the eigenfunctions. Their results indicate that the free-surface instability is caused by the perturbation shear stress at the free surface. Specifically, the instability is due to the competition between the stabilization provided by the hydrostatic pressure and the inertial destabilization. Moreover, Kelly *et al.* (1989) have shown that the perturbation shear stress drives a perturbation vorticity that can either enhance or reduce the deflection of the free surface from its unperturbed position. In fact, it was demonstrated that the stability of the free surface can be determined by examining the phase shift between the disturbance vorticity and the free-surface shape. Specifically, if the maximum value of the disturbance vorticity at the free surface lags the peak of the perturbed free surface, the disturbance vorticity will induce an upward motion of the free surface resulting in growth of the disturbance (see figure 11*a*). Hence, the critical condition for the instability corresponds to a phase difference $\Phi = 0$ and for $\Phi > 0$ the flow is stable.

Huang & Khomami (2000) have examined the mechanism of instability of one-layer viscoelastic flows down an inclined plane based on rigorous energy analysis. Their results indicate that the coupling between the base flow and perturbation velocities

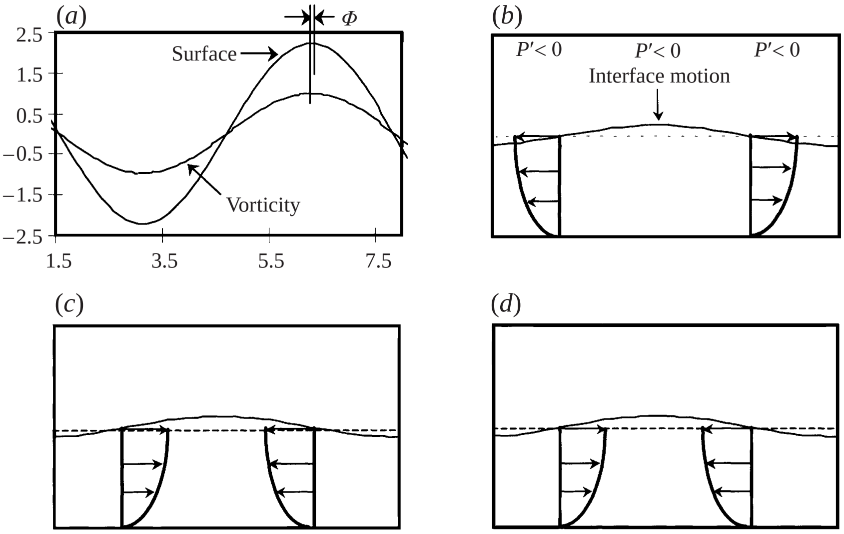

FIGURE 11. The mechanism of instability in one-layer flow down an inclined plane: (*a*) phase difference between the perturbation vorticity and the motion of the free surface in single-layer flow of Newtonian fluid down an inclined plane, $\alpha = 0.05$, $\beta = 45.3°$, $S = 0$; (*b*) hydrostatic pressure contribution; (*c*) the first elastic contribution (i.e. $-We[(\mathrm{d}U/\mathrm{d}y)(\partial u_0/\partial y)]$; (*d*) the second elastic contribution (i.e. $We[u_0(\mathrm{d}^2U/\mathrm{d}y^2)]$.

and the polymeric stresses and their gradients at the free surface is the driving force for the instability. As mentioned above, the mechanism of the instability can also be investigated by a careful examination of the eigenfunctions. For purely elastic instabilities (i.e. $Re = 0$), the linearized equation of motion for the perturbation variables are given by (note in these equations the perturbation pressure has not been eliminated)

$$x\text{-component} \qquad 0 = -\mathrm{i}\alpha p' - \left[\mathrm{i}\alpha F_{xx} + \frac{\partial F_{xy}}{\partial y}\right], \tag{41}$$

$$y\text{-component} \qquad 0 = -\frac{\partial p'}{\partial y} - \left[\mathrm{i}\alpha F_{xy} + \frac{\partial F_{yy}}{\partial y}\right]. \tag{42}$$

The boundary conditions are given by

$$u' = (0) = 0, \quad v'(0) = 0, \quad \text{at y} = 0, \tag{43}$$

$$F_{xy} + \frac{\mathrm{d}\tau_{xy}}{\mathrm{d}y}h' = 0, \quad \text{at y} = 1, \tag{44}$$

$$p' + F_{yy} + \frac{\mathrm{d}p'}{\mathrm{d}y}h' = 0, \quad \text{at y} = 1, \tag{45}$$

$$v' = \mathrm{i}\alpha(U - c)h', \quad \text{at y} = 1. \tag{46}$$

Solving the above set of equations in the limit of long-wave disturbances, (i.e. $\alpha \rightarrow 0$) one obtains

$$F^0_{xy} = -\frac{\partial u_0}{\partial y}, \quad F^0_{yy} = 0, \tag{47}$$

$$\frac{\partial^2 u_0}{\partial y^2} = 0, \quad u_0(0) = 0, \quad \frac{\partial u_0(1)}{\partial y} = 2, \tag{48}$$

$$\frac{\partial p_0}{\partial y} = 0, \quad p_0(1) = 2\cot\beta, \tag{49}$$

$$u_0(y) = 2y, \quad v_0 = 0, \quad p_0(y) = 2\cot\beta. \tag{50}$$

Clearly one needs to proceed to the next-order expansions in α in order to determine the stability of the system. At order α, the following equations are obtained:

$$\frac{\partial v_1}{\partial y} + \mathrm{i}u_0 = 0, \quad v_1(0) = 0, \quad c_0 = U(1) + \mathrm{i}v_1(1), \tag{51}$$

$$v_1(y) = -\mathrm{i}y^2, \qquad c_0 = 2, \tag{52}$$

$$0 = -\mathrm{i}p_0 - \mathrm{i}F^0_{xx} - \frac{\partial F^1_{xy}}{\partial y}, \tag{53}$$

$$F^0_{xx} = -2F^0_{xy}\left(\frac{\mathrm{d}U}{\mathrm{d}y}\right) - 2\tau_{xy}\frac{\partial u_0}{\partial y}, \tag{54}$$

$$F^1_{xy} = -\frac{\partial u_1}{\partial y} + \mathrm{i}\left\{2WeF^1_{yy}\left(\frac{\mathrm{d}U}{\mathrm{d}y}\right) - (U - c_0)F^0_{xy} - We\frac{\mathrm{d}\tau_{xy}}{\mathrm{d}y}v_1\right\}, \tag{55}$$

$$\frac{\partial^2 u_1}{\partial y^2} = \mathrm{i}p_0 - \mathrm{i}We\left[\left(\frac{\mathrm{d}U}{\mathrm{d}y}\right)\left(\frac{\partial u_0}{\partial y}\right) - u_0\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right], \tag{56}$$

$$u_1(0) = 0, \quad \frac{\partial u_1}{\partial y} = 0, \tag{57}$$

$$u_1(y) = \mathrm{i}p_0\left(\tfrac{1}{2}y^2 - y\right) - \mathrm{i}We(2y^2 - 4y). \tag{58}$$

Once again, one needs to proceed to a higher-order expansion to determine the critical conditions for linear stability. At order α^2, the following equations are obtained:

$$\frac{\partial v_2}{\partial y} + \mathrm{i}u_1 = 0, \quad v_2(0) = 0, \quad c_1 = \mathrm{i}v_2(1), \tag{59}$$

$$v_2(y) = \cot\beta\left(\tfrac{1}{3}y^3 - y^2\right) - \tfrac{2}{3}We\,y^3 + 2We\,y^2, \tag{60}$$

$$c_1 = \mathrm{i}(-\tfrac{2}{3}\cot\beta + \tfrac{4}{3}We). \tag{61}$$

Therefore, using equation (61), the We_{crit} above which the system is linearly unstable can be determined:

$$We_{crit} = \tfrac{1}{2}\cot\beta. \tag{62}$$

The results of the analysis at $O(\alpha)$ shows that the balance between the hydrostatic pressure due to the displacement of the interface and elastic forces determine the sign of the perturbation velocity in the mean flow direction (see equation (58)). In fact, an examination of equation (56) shows that there are two separate elastic contributions. The first elastic contribution, i.e. $-We\,[(\mathrm{d}U/\mathrm{d}y)(\partial u_0/\partial y)]$, is due to the coupling between the base-flow shear stress and the perturbation velocity gradient. Since $(\mathrm{d}U/\mathrm{d}y)$ and $(\partial u_0/\partial y)$ are always positive in the domain, this elastic term is always negative and is proportional to We. The second elastic contribution, i.e.

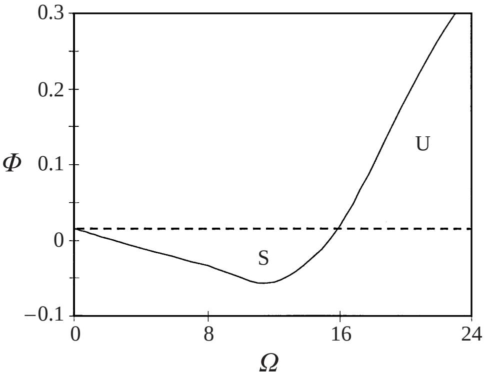

FIGURE 12. The phase difference between the perturbation vorticity and the motion of the free surface (i.e. Φ) as a function of modulation frequency for one-layer UCM flow down an inclined plane, $\beta = 45.3°$, $Re = 0.01$, $We = 0.5$, $\alpha = 0.05$, $\Lambda = 3$, $S = 0$.

$We(u_0\,\mathrm{d}^2U/\mathrm{d}y^2)$, is due to the coupling between the perturbation velocity and the base-flow shear stress gradient. Since $(\mathrm{d}^2U/\mathrm{d}y^2)$ is always negative and u_0 is always positive, this term is also always negative in the domain. Hence, if one considers a perturbed interface, the hydrostatic pressure term tends to push fluid away from under the crest (see figure 11*b*) while the elastic terms tend to introduce fluid under the crest (see figures 11*c* and 11*d*). Therefore, when the contribution of the elastic terms that are proportional to *We* exceed a certain critical value the flow becomes unstable.

The magnitude of each term can be examined by measuring the power associated with it. The average power is defined

$$Z = \left\{\int_0^1 p_0 - We\left[\left(\frac{\mathrm{d}U}{\mathrm{d}y}\right)\left(\frac{\partial u_0}{\partial y}\right) - u_0\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right]\right\}\mathrm{d}y = 2\cot\beta - 4We. \qquad (63)$$

Hence, in the limit of long waves, elastic effects are always destabilizing and proportional to *We*, while hydrostatic pressure is stabilizing. In addition, the first (i.e. $-We[(\mathrm{d}U/\mathrm{d}y)(\partial u_0/\partial y)]$), and the second (i.e. $We(u_0\mathrm{d}^2U/\mathrm{d}y^2)$), elastic contributions have the same magnitude. It should be noted that the results of the above analysis is in full agreement with the rigorous energy analysis result of Huang & Khomami (2000).

To examine the effect of dynamic modulation on the mechanism of stability of both Newtonian and viscoelastic flows down an inclined plane we have performed an energy analysis using the procedure developed in our earlier study. The details of the energy analysis are given in our earlier paper (Huang & Khomami 2000), hence they are not reproduced here.

The energy analysis has been performed based on the Floquet exponents. The results of these analyses show that the mechanism of the instability in both Newtonian and viscoelastic flows is unchanged in the presence of dynamic modulation. That is the main driving force for the Newtonian inertial instability is the perturbation shear stresses at the free surface while the elastic destabilization mechanism in the limit of vanishing *Re* is due to the coupling between the base flow and the perturbation velocity and stresses and their gradients at the free surface.

Although, dynamic modulation does not alter the mechanism of the instability it clearly influences the onset conditions for the instability. To illustrate this point, one can express the perturbation velocities generated at the free surface in terms of perturbation vorticities. In turn, as shown by Kelley *et al.* (1989) the phase difference between the perturbation vorticity at the free surface and the motion of the free surface can be used to illustrate the effect of dynamic modulation on the stability characteristics of the flow. Figure 12 depicts the phase difference between the perturbation vorticity and the free-surface motion. Clearly, the stabilizing or destabilizing influence of dynamic modulation is a direct consequence of altering the plane difference between the motion of the free surface and the perturbation vorticity at the free surface that exists in the absence of dynamic modulation. Specifically, when $\Omega < 16$ dynamic modulation reduces Φ, hence the flow is stabilized while for $\Omega > 16$ the opposite trend is observed.

4. Conclusions

In this study, the effect of dynamic modulation on the stability characteristics of one-layer Newtonian and viscoelastic fluids down an inclined plane has been theoretically examined. It has been demonstrated that parallel superposition of modulation can be used to stabilize or destabilize this class of flows. Specifically, it is shown that the effect of dynamic modulation on the stability characteristics of viscoelastic flows down an inclined plane is more pronounced than that of Newtonian flows with $O(1)\,Re$.

The dependence of the most-dominant modulation frequencies on Re and We has also been examined. It has been shown that for Newtonian flows (in the limit of long waves) low-frequency modulations are destabilizing and the most dominant frequency scales with $1/Re$. However, for viscoelastic flows in the limit of vanishing Re, it has been shown that low-frequency modulations are stabilizing and the most dominant modulation frequency scales with $1/We$. In the case of viscoelastic flows with finite Re, it is observed that the most destabilizing frequencies scale with $1/Re$ while the most stabilizing frequencies scale with $1/ReWe$. These results clearly depict the competition between the elastic and inertial forces in the presence of dynamic modulation. Moreover, these scalings are independent of the inclination angle and the wavenumber (in the limit of long and $O(1)$ waves). In addition, it has been shown that for viscoelastic flows the stabilizing or destabilizing influence of dynamic modulation is proportional to the amplitude of the modulation. However, relatively large modulation amplitudes are required to achieve significant stabilization/destabilization with the parallel superposition modulation technique.

Finally, it is shown that the mechanism of both purely elastic and inertial instabilities in flows down inclined planes is unchanged in the presence of dynamic modulation. In fact, our analyses indicate that modulations of the solid surface greatly influence the frequency of the free-surface wave as well as its phase difference with the disturbance vorticity at the free surface.

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